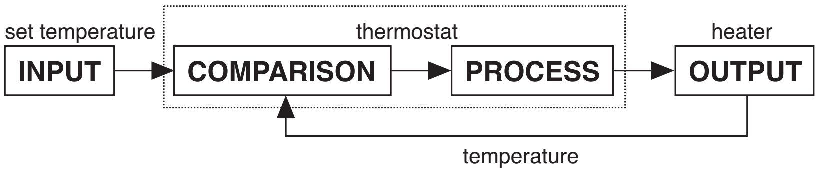
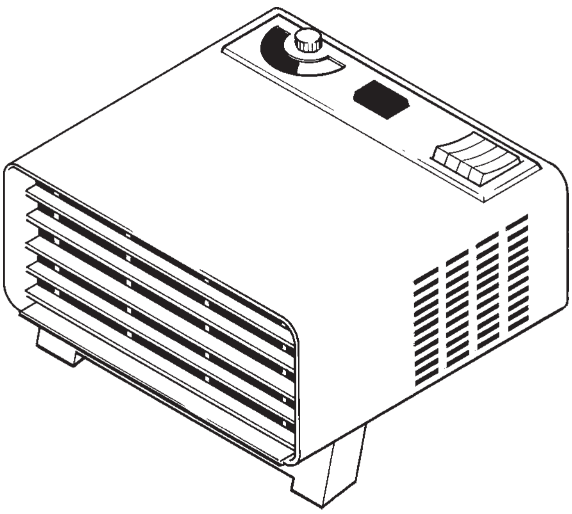
MATHEMATICAL REPRESENTATION OF CONTROL SYSTEMS

INTRODUCTION

Control systems are often drawn using block diagrams such as that shown below;





Lines
The lines in these diagrams represent **information** in the control system. One way to represent this information is as **variables** in the system; that is, values that can change as the system is working and that can be measured.

Examples of variables in a control system are;

- motor speed (r.p.m)
- heater power (Watts)
- temperature (°C)
- level of light (Lux)
- the level of electrical signals (volts)

Blocks
The blocks represent things that either **combine** variables or **change** variables.

Things that **combine** variables include thermostats and electronic difference amplifiers. Things that change variables include electronic amplifiers and all sensors and actuators. For example a light sensor turns a light level into an electronic signal, a motor turns an electronic signal into rotary movement.

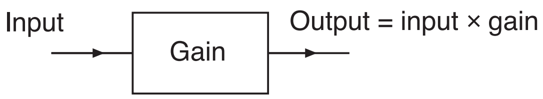
Blocks can be used to represent quite complex systems. For example a complicated electronic circuit could be described by a block simply labelled as 'Gain'.

Block diagrams are used by Engineers because they make it easier to picture what is happening in a control system. They also help Engineers to build up a description of the control system using mathematical language.

Using Maths to Describe Control Systems

Most of the control systems in this module can be described using just two basic blocks;

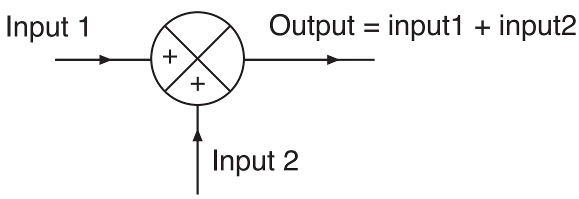
1) A Multiplying Block
This takes a value (the **input**) and multiplies it by a set amount. This multiplying factor is called the **gain**;



Eg; Input = 6
Gain = 2.5
Output = 15

The value of the gain can be bigger than 1 or smaller than and positive or negative. It will all depend on what the control system needs to do.

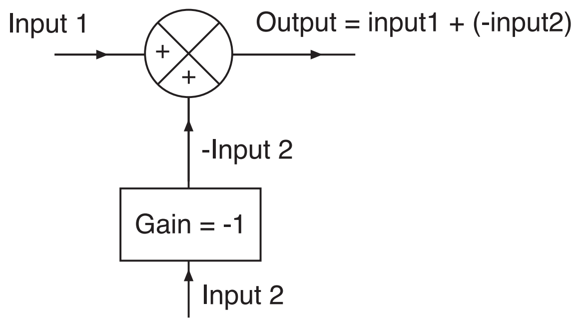
2) A Summing Block
This has two inputs and adds them together;



Eg; Input1 = 10
Input2 = 1.5
Output = 11.5

Sometimes a block is needed that takes one input away from another one instead of adding them together.

This can be done by multiplying one of the inputs by -1 first;



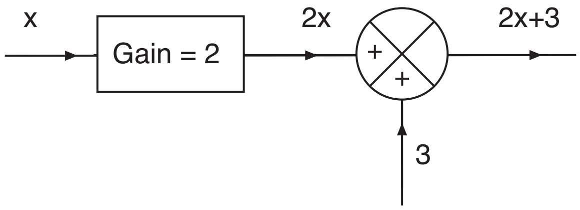
Eg; Input1 = 10
Input2 = 1.5
Output = 8.5

Remember that a block diagram can be used to describe **any** kind of control system - not just electronic control systems. In industry a lot of control is performed by pneumatic systems, hydraulic systems and by various mechanical systems. All of these can be described by block diagrams as well.

In mathematics these block diagrams are often called **Number Machines**. Although the name is different they work in the same way. The following sections of work will help you understand how these block diagrams, or number machines, can be used.

SECTION 1. BLOCK DIAGRAMS AND FORMULAE

Here is a simple number machine with an input, called ‘x’ and an output, 2x+3;



If x=7;

7 ⟶ 14 ⟶ 17

so 2x+3 = 17.

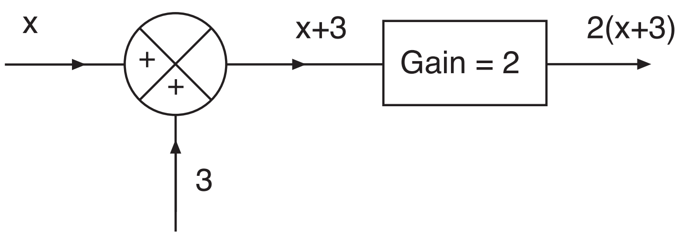
To Do

- Suppose x = 3. What is 2x+3?
- Write down other values for x in a table like this;

x	7	3	2				
2x+3	17	?	?				

- Repeat this for a number machine that has an output of 5x-7.
  Draw the number machine for this output.
- Draw up a similar table and number machine for $\frac{x}{3}$ - 2

If you swap the boxes around you get a different formula;



What difference does this make?

If x=7;

7 ⟶ 10 ⟶ 20

2(x+3) = 20.

To Do

- If x = 3, what is 2(x+3)?
- Use the same values for x that you used above to fill in a table like this;

x	7	3	2				
2(x+3)	20	?	?				

- Repeat this for a number machine that has an output of 5x-7.

  Draw the number machine for this output.

- **Problem**
  Can 2x+3 ever be bigger than 2(x+3)? Investigate this.

  Helping Hints

  a) Collect some results;

Number tried	2x+3	2(x+3)	Is 2x+3 bigger?
2	=7	=10	No

  b) Be systematic.
  c) Put results in a table.
  d) Look for a pattern. Predict a result.
  e) Try out new cases to check your ideas.
  f) Can you explain your findings?
  g) Draw graphs or solve equations to prove your ideas.

  If you needed to use the hints, try this;

  Sometimes 3(x-1)/5 is more than x. Investigate. (It may help to draw the number machine for this formula.)

- Draw up the number machines for the following formulae;

  1. Input = x.
  Output = 3x-7.

  So if x=5, 3x-7=______?

  2. Input = x.
  Output = 3(x-7).

  So if x=510, 3(x-7)=______?

  3. Input = x.
  Output = (2x+3)/5.

  So if x=16, (2x+3)/5=______?

  4. Input = x.
  Output = 5(x+3)-2.

  So if x=16, 5(x+3)-2=______?

- **Problem**.
  The formula linking ˚C (degrees Celsius) to ˚F (degrees Fahrenheit) is

$$°F = \frac{9°C}{5} + 32$$

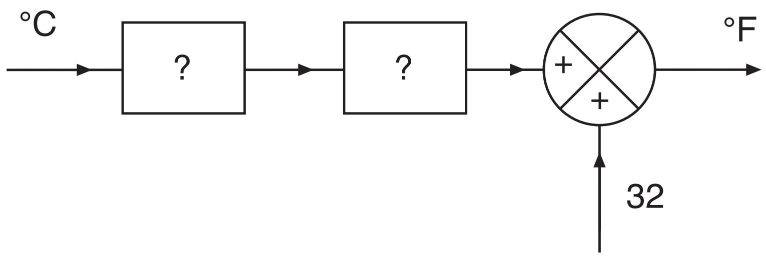
Make a number machine for this formula.

Use the number machine to help you make;

- A ‘ready reckoner’ table for converting between the two temperature scales
- A conversion graph

Helping Hints.

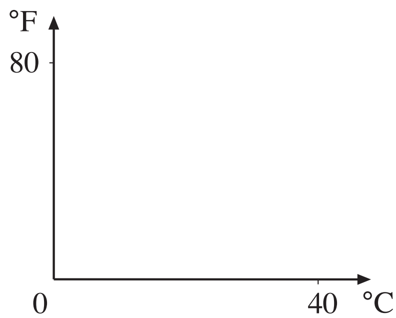
a) The number machine will need three function blocks;



b) A possible table

°C	0	5	10	15	20	25	
°F	32						

c) A possible graph

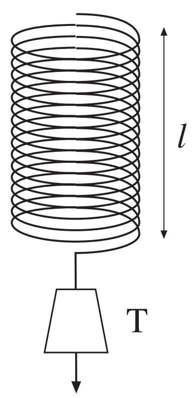


If you needed to use the hints, try this;

In an experiment some engineers have found a link between the length of a particular spring and the load hanging on it. The formula linking the length and load is;

$$T= 100(l\text{-}15)$$

Where **T is the load** on the spring (in Newtons) and *l* **is the length** of the spring (in metres). If the spring is used in a spring balance then the length of the spring can tell you the load hanging on the balance;



Make a number machine for the formula.
Use the number machine to help you make;

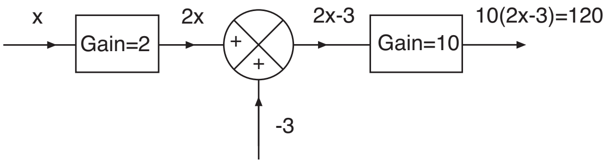
- A 'ready reckoner' table for converting from length to weight
- A conversion graph between length to weight

SECTION 2. SOLVING EQUATIONS WITH BLOCK DIAGRAMS

Sometimes engineers have a control system already set up and know what its output should be. The problem for them then is to work out what input is needed to give the right output.

For example when a robot arm is being controlled the 'output' is the position that the arm should be in. An engineer needs to work backwards from this bit of information to work out what input should be go into the system to get this correct output. Block diagrams, or number machines, can help solve this kind of problem.

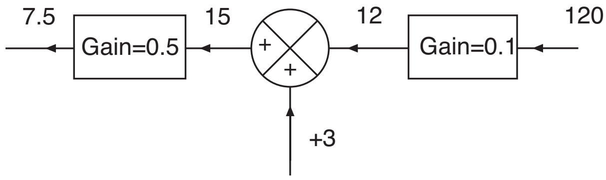
For example, in the equation 10(2x-3) = 120, '120' is the output. What value of x is needed to get this output?



You can solve this problem by using '**inverse**' machines;

The inverse of	is
multiply	divide
divide	multiply
add	subtract
subtract	add

So the inverse machine for the block diagram is;



So the required input is **<u>7.5</u>**.

The formal way of writing this down is;

10(2x-3) = 120

Divide both sides by 10,

2x-3 = 12

add 3 to both sides,

2x = 15

Divide both sides by 2,

x = 7.5

To Do

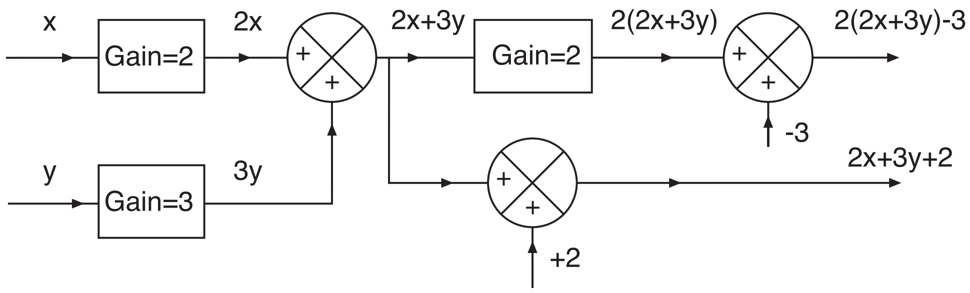
- Use number machines to work out the solutions to the following equations (that is, find the value of 'x');
  - (7x-3) = 32
  - 2(x+3) = 12
  - 2x+3 = 12
  - 5(x+3)-2 = 15.5
  - (2x+3)/5 = 7
  - 3(x-7)+21 = 3

Helping Hints

First draw the number machine for each formula. Then use this to draw the inverse machine.

SECTION 3. DIAGRAMS WITH MORE THAN ONE INPUT

Number machines and block diagrams can have more than one input and output;



In this number machine, if **x = 10** and **y = 1** then;

2x+3y +2 = 20+3+2 = <u>*25*</u>

and

2(2x+3y)-3 = 2(20+3)-3 = <u>*43*</u>

Take another example; if **x = 7** and **y = 3** then;

2x+3y +2 = 14+9+2 = <u>*25*</u>

and

2(2x+3y)-3 = 2(14+9)-3 = <u>*43*</u>

............so, if you have more than one input, you can get the same output from different values of the inputs. This is true of many control systems.

To Do

- Can you find other values of x and y that also give 2x+3y +2 = <u>25.</u>

  Do all of these pairs of values also give 2(2x+3y)-3 = <u>43</u>
  (Hint; try x = 4........)

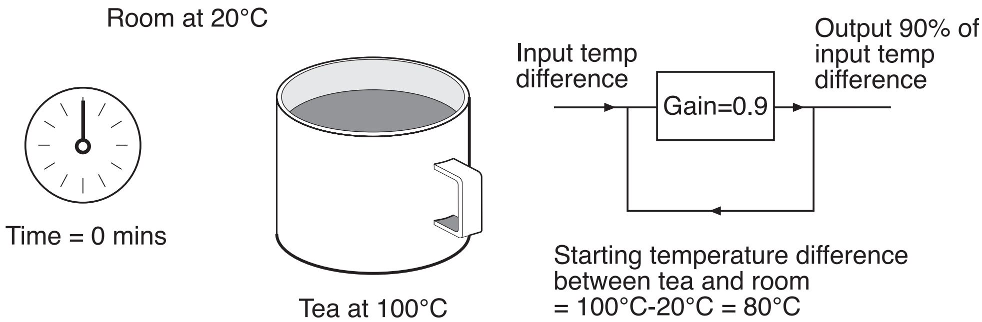
- Make number machines for the following;

  - 2x+5y+7
  - 2(x+2y)+y+7
  - 2(x+y)+3y+7
  - 5(x+y)-3x+7

- Now see what happens when you try **x=10** and **y=4** in each of these. Investigate this.

SECTION 4. SOLVING PROBLEMS WITH NUMBER MACHINES

A. Cooling down
A cup of tea is cooling down. The difference in temperature between it and the room goes down by 10% every minute;



So the starting temperature difference, T_0 = 80 ˚C.

After one minute the difference is 90% of this;

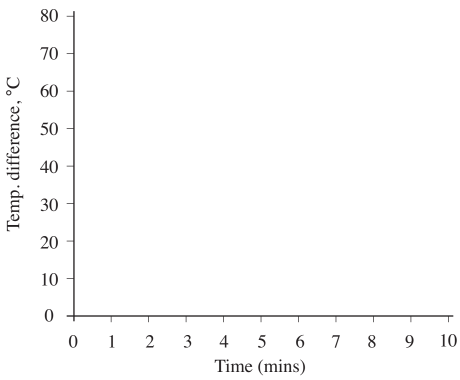
$$T_1 = 80 \text{ x} 0.9\ ˚C = 72˚C$$

then after two minutes,

$$T_2 = 72 \text{ x} 0.9\ ˚C = 64.8˚C$$

To Do

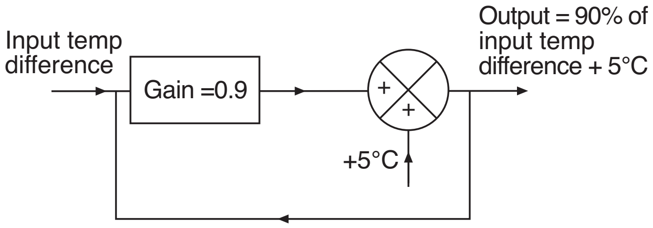
- After three minutes, T_3 is 90% of 64.8, and so on. Calculate T_3, T_4.......up to T_{10}, Plot the values as a graph;

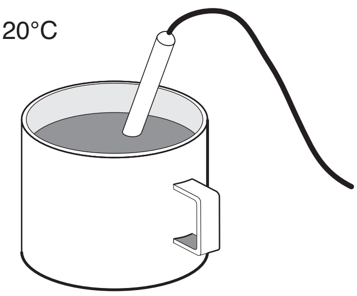


- The *actual* temperature of the tea is the temperature difference (T_x) plus 20˚C. Plot a graph of the actual temperature of the tea over 10 minutes.

- Take some readings from a real cup of tea. Investigate your results.

B. Warming up
An immersion heater is placed in the tea. This warms the tea up by a fixed amount of 5˚C each minute;





Now;

$$T_1 = (80 \text{ x}0.9)+5 \text{ ˚C} = 77\text{˚C}$$

To Do

- Calculate T_2, T_3.......up to T_{10}, Plot the values as a graph as before.
- Investigate ;
  - Different starting temperatures.
  - Different heaters (eg adding 10˚C per minute).
- Think about the results you have; do they make sense? (What is the boiling point of water.)
  Try taking some real readings to see what happens. Look particularly at temperatures near boiling point.